Some Mathematical Coincidences

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INTRODUCTION. The field of mathematics is very wide and it is not easy to predict what happens next, but I can tell you it is alive and well. Two general trends are obvious and will surely persist. In its pure aspect, the subject has changed, much for the better I think, by moving to more concrete problems. In both its pure and applied aspects, an equally beneficial shift to non-linear problems can be seen. Most mathematical questions suggested by Nature are genuinely non-linear, meaning very roughly that the result is not proportional to the cause, but varies with it as the square or the cube, or in some more complicated way. The study of such questions is still, after two or three hundred years, in its infancy. Only a few of the simplest examples are understood in any really satisfactory way. I believe this direction will be a principal theme in the future. To make concrete these vague remarks would require examples described in some detail, hard to do in a little space, so I have chosen a more historical route.

Mathematics is a language, providing a mode of very precise, compactly expressed, rapid thought. That is a commonplace. What is less generally known is that it also permits you to recognize unexpected real connections and/or mere parallels between different aspects of the natural world which, from a so-to-say philosophical point of view, have *nothing to do with each other*. So I thought I could describe three instances of this peculiar business, one old, from 1820 or so (gravity and electrostatics); one quite recent from 1967 (solitary ocean waves & quantum mechanics); and one brand new, but with old roots from the 1800's (primes and atomic nuclei). I hope these three little stories will illustrate how ideas from the natural world circulate in and out of mathematics, much to the advantage of the latter, but *slowly*, on a very long time scale.

Here, I ought to say that the expression of natural phenomena in the mathematical way can be a mixed blessing, not always favorably received. Mostly, such a mathematical description ignores “small” effects and its validity is cannot be expected in too wide a variety of circumstances, so it must be reckoned a mistake if the mathematician takes it too seriously, way beyond the original intent. Goethe said: “Mathematicians are like the French: You tell them something, they put it into their own language, and then it means something completely different”[1]. But mistakes can be avoided with a little care, and then the mathematical language works surprisingly well, sometimes in ways unforeseen; see E. Wigner: The unreasonable effectiveness of mathematics in the natural sciences. *Comm. Pure & Appl. Math.* **13** (1–14) 1960.

GRAVITY & ELECTROSTATICS. The mathematical description of gravity goes back, of course, to Newton and his *force* = *mass* × *acceleration*, upon which all classical mechanics is founded. His derivation of Kepler’s three laws of planetary motion:

1) that the planets move in elliptical orbits having one focus at the sun;
2) that the radius vector from the planet to the sun sweeps out equal areas in equal times;
3) that the period of motion is proportional to the major axis of the ellipse, raised to the power 3/2

was a stunning event, quite comparable in its day to the discoveries of Einstein, of Heisenberg and Schrödinger, and of Crick and Watson in the century just past. Newton’s law of gravitation supplies the f in $f = ma$: it states that the (attractive) force acting between two masses m_1 and m_2 is $f = gm_1m_2/r^2$ in which r is the distance between the two bodies and g is the universal gravitational constant (which can be measured). The rest is mathematics, *i.e.* solving the equations of motion so produced to obtain Kepler’s 1), 2), 3). This dates from Newton’s presentation of his *Principia* to the Royal Society in 1686.

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[1] M. Gebhardt: *Goethe als Phyiker.* G. Grotésche Verlags-buchandlung, Berlin 1932.

Now the story jumps ahead to 1820 or so, to Faraday's great researches on electricity and magnetism[2], formalized in part by Coulomb's law: that the (repulsive) force acting between two like electrical charges e_1 and e_2 is $f = ce_1e_2/r^2$, with a new (electrostatic) constant c different from g, but in every other respect the *same* as Newton's law. But what has electricity to do with gravity? Is it some king of anti-gravity? Not at all! It is only the mathematics that is like. This is peculiar but there it is. Notice the slow time scale: more than 100 years.

SOLITARY WAVES AND QUANTUM MECHANICS. The late 18th and the 19th centuries supplied us with a description of how water flows, satisfactory in principle if not completely so in practice; in particular, the formation and advance of ocean waves was much studied by Geo. Stokes and others. Naturally, it all starts from Newton's $f = ma$: the m is the mass density of water, taken as constant by virtue of incompressibility; the a explains itself; the f is due to the gradient of the pressure: If you sit still in the water the pressure is all around you; it does not push you either one way or another. But if you move *with* the flow from, *e.g.* low to high pressure, the gradient of the pressure is opposed to this. Newton's $f = ma$, written out with this picture in mind is the equation of (incompressible) fluid flow in Euler's form (1761):

$$\frac{\partial v}{\partial t} + (v \cdot \text{ grad})v + \text{ grad } p = 0.$$

It is of extreme difficulty and only very special cases and/or approximations have been fully understood. Stokes, himself, studied the existence and permanence of solitary waves, so-called, traveling vast oceanic distances without change of shape. Think of a *Tsunami* (tidal wave) and, for a description of huge waves at sea, read Sebastian Junger's terrifying book *A Perfect Storm.*

Now if the water is shallow and the wave is long, the surface velocity $v(t, x)$ near its leading edge, as a function of the time $t \geq 0$ and the distance x in the direction of motion measured from some fixed point of reference, develops according to the equation of Korteweg-de Vries

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[2] Incidentally, Faraday did not think mathematically, but rather in geometrical pictures. It was only in 1861 that Clerk Maxwell gave a full mathematical discussion.

(1896):

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{\partial^3 v}{\partial x^3},$$

originally invented to describe long waves in a shipping canal. Here, the $v \times \partial v/\partial x$ to the left tends to produce "shocks", somewhat as when a plane passes through the sound barrier, while the $\partial^3 v/\partial x^3$ to the right makes wave "packets" of different frequencies to move at different speeds, proportional to k^2. This effect of "dispersion" competes with shocks; in fact, it prevents them from forming at all, so nothing bad happens. But how to solve such an equation with its physically essential but mathematically unpleasant non-linearity $v \times \partial v/\partial x$?

Let me make an analogy. The equation $dx/dt = 1$ is easy to solve $[x(t) = x(0) + t]$, but $dx/dt = \sqrt{1 - x^2}$ is not, at least if you don't know anything. But if you remember your trigometrical functions and have to wit to substitute $x = \sin\theta$, then $dx/dt = \cos\theta \times d\theta/dt$ is equated with $\sqrt{1 - \sin^2\theta} = \cos\theta$, which is to say $d\theta/dt = 1$, and you are back in business. This is all too simple-seeming, but remarkably enough there is a similar substitution, not now with only one degree of freedom, but in ∞-dimensional "function" space, replacing v by a new function w, subject to KdV with the non-linearity crossed out: $\partial w/\partial t = \partial^3 w/\partial x^3$. This is easily solved for w and v is reconstructed from that, by inverting the substitution. But where does this substitution come from?

Here, I cannot omit to quote C.G.J. Jacobi from his great *Vorlesungen über Dynamik.* Ges. Werke, Band, Reimer, Berlin, 1884, where he says (roughly) "We are supposed to be solving differential equations, but of course we don't know how to do it unless, by luck, we hit upon some clever substitution which simplifies our task. And as there is no *rule* for finding this, so we take some substitution which seems attractive on quite different grounds and look about for differential equations which can be solved by it." Quite backwards you may say, but then he goes on to do a lot of beautiful mechanics in just this way. But now back to KdV and to the question of the "correct" substitution for solving it.

Well, it came as a complete surprise that a one-dimensional of Heisenberg's description of the scattering of a beam of electrons off an atomic nucleus is just what the doctor ordered. This from 1967[3]. I will not explain what hints prompted this surmise, but here are the facts.

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[3] M. Kruskal *et al.*: Method for solving the KdV equation. *Phys. Rev. Letters* **19** (1967) 1095–1097.

You start from a one-dimensional version of Schrödinger's equation: $H\psi = -d^2\psi/dx^2 + v(x)\psi = E\psi$. Here $v(x)$ is the classical potential energy of an electron at distance x from the proton or whatever; $-d^2/dx^2$ plays the role of kinetic energy; and the number E is interpreted as total energy when the system is in the "pure state" ψ—— so much for the lingo. Now a beam coming in from $x = +\infty$ infringes on the proton: Part of it is reflected back and part of it is transmitted in proportions and with phase shifts depending upon the wave number $k = \pm\sqrt{E}$; and it is a fact, of great practical importance in the real 3-dimensional setting, that v can be reconstructed from measurements of these reflection and transmission coefficients for sufficiently many values of the wave number.

Now back to water. Replace $v(x)$ in H by the fluid velocity $v(t, x)$ moving with time. Then, remarkable to relate, the transmission coefficient does not move at all, but the (moving) reflection coefficient w solves KdV with the non-linearity crossed out: *i.e.* up to a constant multiplier of no importance, $\partial w/\partial t = \partial^3 w/\partial x^3$, as was wanted. But who says I can view the KdV velocity as a quantum-mechanical potential, which is an entirely different animal, or, more bluntly, what have solitary waves to do with electrons, any how? Obviously, *nothing at all.* But there it is. It works. And it's a free country as a New Hampshire neighbor of mine liked to say when somebody did something really silly.

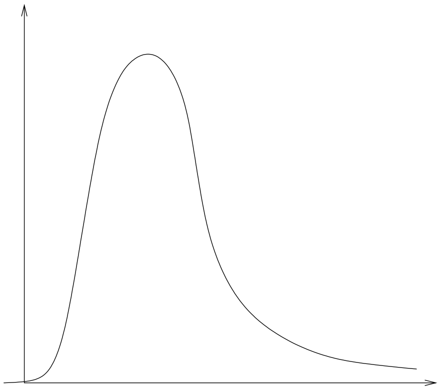
PRIMES AND ATOMIC NUCLEI. The prime numbers $p = 2, 3, 5, 7, 11, 13$ *etc.* and their very erratic distribution among the whole numbers $n = 1, 2, 3, 4, 5$ *etc.* have fascinated mathematicians ever since Euclid who proved there must be infinitely many of them. That is from very long ago. Nowadays, the most up-to-date high security codes are made to depend upon the prime factorization of very large numbers. So there is more than one reason to be interested in primes.

The present story is, in part, quite new and its ending is not yet known. It begins long ago with Gauss, around 1800. He made himself a big table of primes from which he guessed that the number of primes $p \leq n$ is roughly $n/\log n$ for large n. Riemann tried to prove this in 1860 or so. He even proposed a sharp bound to the error of $\sqrt{n}$, more or less, depending upon his still unproven hypothesis that, apart from a series of known irrelevant exceptions, all the roots of a certain auxiliary function ("zeta" as it is called) lie on the line $[\frac{1}{2} + \sqrt{-1}x : -\infty < x < \infty]$ in the complex plane. Gauss' law, without the fine error term, was finally proved about 1890 with the help of pretty heavy mathematical machinery involving zeta, and again, about 1955, in an "elementary" way.[4] That is the first part of the story.

Now at about the same time as this "elementary" proof, Wigner proposed that the distribution of energy levels in complicated atomic nuclei might be quite faithfully reproduced by the statistics of eigenvalues in certain ensembles of $(d \times d)$ random matrices in the limit of large dimension $(d = \infty)$. If these notions are unfamiliar, don't worry. It would take a while to explain, and I think I can make my point anyhow.

The ensembles employed are of several different kinds; the simplest is the unitary ensemble, $U(d)$ as it is called. The eigenvalues of such a $d \times d$ unitary matrix may be pictured as points in the unit circle; there are d of them, situated at angles $0 \le \theta_1 \le \theta_2 \le \theta_3$ *etc.* $\le 2\pi$, and their joint distribution can be calculated quite explicitly. Now, for large d; the spacing between two consecutive angles is about $2\pi/d$ on average, and if you scale it up by the reciprocal $d/2\pi$, you can compute the probability density of the scaled spacing for $d = \infty$. The picture will give you a very rough idea of how it looks: more or less bell-shaped but with a long tail to the right. But how can this have *anything* to do with primes?

FIGURE



The answer is most surprising. To tell you what it is, I go back to "zeta" and Riemann's idea that all its interesting roots lie on the line $[\frac{1}{2} - \sqrt{-1}x : -\infty < x < \infty]$. The fact is that billions (!) of these roots have been computed and they all fall right on the line. How

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[4] Here, "elementary" does not mean easy; it is only that no machinery from outside arithmetic is used.

that can be known by mere, error-prone computation is another story I will not tell here. Now comes the punch-line, *to wit*, that to the naked eye, the empirical histogram for the (properly scaled) spacings between consecutive roots of zeta is indistinguishable from the spacing density for the unitary ensemble seen above. Why, nobody knows, but it has to be believed that something deep is happening here. Perhaps we'll find out before too long.

CONCLUSION. I could go on: for example, I might tell you that the machinery developed for KdV has had ramifications in the area of "projective curves", one of the purest regions of mathematics that exists, and what have solitary waves to do with that? or I might explain that there also connections between KdV and matrix ensembles, and with optics, too, and with the geometry of surfaces of negative curvature, and so forth; but I hope you got the idea already. Here are three striking coincidences which have no business being there, nor could they have come to light without the mathematical language to express them. It is, to say the least, *very odd.* Goethe said: "Be hold and mighty forces will come to your aid".

A FEW REFERENCES. For gravity, electrostatics, and indeed for any aspect of physics, old or new, *the* best place to look is R. Feynman: *Lectures in Physics* (3 vols.). Addison-Wesley, Reading, MA 1963–65. For more information about "KdV and all that", see G. Lamb: *Elements of Soliton Theory.* John Wiley, New York, 1980; see also M. Kruskal: Non-linear wave equations. *Lect. Notes in Physics* **38** (1975) 310–354. For prime numbers, Riemann, and the unitary ensemble, see J.B. Conrey: The Riemann hypothesis. *Notices Amer. Math. Soc.* **50** (341–353) 2003. He has nice pictures, too.

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